

The Ellipsoid of Concentration

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ABSTRACT

The material given by Cramer on the ellipse of concentration is here presented in matrix form leading to a simplification of some of the results.

If X is an n -dimensional random variable with mean μ and variance-covariance matrix Σ , then the ellipsoid of concentration is given by the quadratic form

$$(Y-\mu)\Sigma^{-1}(Y-\mu)' = n+2 \quad .$$

A uniform distribution over the domain bounded by this ellipsoid then has the same first and second order moments as the given distribution.

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The purpose of this note is to present, in matrix notation, the material given by Cramer¹ on the ellipse of concentration. The use of matrix notation considerably simplifies the derivation, especially in the cases of more than two variables.

Let X be a random variable with mean μ and variance σ^2 . If X' is another random variable, with a uniform distribution over the interval $(\mu - \sigma\sqrt{3}, \mu + \sigma\sqrt{3})$, then X' has the same mean and variance as X . The interval $(\mu - \sigma\sqrt{3}, \mu + \sigma\sqrt{3})$ may thus be taken as a geometrical representation of the concentration of the distribution of X about μ .

In the case of a bivariate distribution with mean (μ_1, μ_2) we wish to find an analogous geometrical representation of the concentration about (μ_1, μ_2) . For this we wish to find a curve enclosing (μ_1, μ_2) such that a uniform distribution over the area enclosed by this curve will have the same first and second order moments as a given bivariate distribution. In this general form the problem is undetermined so we will limit ourselves to finding an ellipse having the required property.

Let the given bivariate distribution have mean $(0,0)$ and variance-covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} .$$

Consider the non-negative quadratic form

$$q(X_1, X_2) = (X_1, X_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$= XAX'$$

¹ Cramer, H. 1945. Mathematical Methods of Statistics. Sections 21.10 and 22.7. Princeton University Press, Princeton.

The area enclosed by the ellipse $q(X_1, X_2) = c^2$ is

$$\frac{\pi c^2}{\det A}.$$

If unit mass is uniformly distributed over this ellipse then the mean of this distribution will be $(0,0)$ and its variance-covariance matrix will be

$$\frac{c^2}{4 \det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{12} & a_{11} \end{pmatrix} = \frac{c^2}{4} A^{-1}$$

It is required to determine c and the a_{ij} such that

$$\frac{c^2}{4} A^{-1} = I$$

This can obviously be done by taking $c^2 = 4$ and

$$\begin{aligned} A &= \Sigma^{-1} \\ &= \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}^{-1} \\ &= \frac{1}{1-\rho^2} \begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1 \sigma_2} \\ -\frac{\rho}{\sigma_1 \sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix} \end{aligned}$$

In the general case of an arbitrary mean (μ_1, μ_2) it thus follows that a uniform distribution over the area enclosed by the ellipse

$$\frac{1}{1-\rho^2} \left(\frac{(X-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(X-\mu_1)(Y-\mu_2)}{\sigma_1 \sigma_2} + \frac{(Y-\mu_2)^2}{\sigma_2^2} \right) = 4$$

has the same first and second order moments as the given distribution. This ellipse is called the ellipse of concentration.

In matrix form the ellipse is

$$(X_1 - \mu_1, X_2 - \mu_2) \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \end{pmatrix} = 4$$

In general let $X = (X_1, \dots, X_n)$ be an n dimensional random variable with mean $\mu = (0, \dots, 0)$ and variance-covariance matrix Φ . Consider the non-negative quadratic form

$$q(X_1, \dots, X_n) = XAX' \quad .$$

If a unit mass is uniformly distributed over the domain bounded by the n -dimensional ellipsoid $q(X_1, \dots, X_n) = c^2$ then the mean of this distribution will be $(0, \dots, 0)$ and its variance-covariance matrix will be

$$\frac{c^2}{n+2} A^{-1}$$

It is now required to choose c and the elements of A such that

$$\frac{c^2}{n+2} A^{-1} = \Phi \quad .$$

This can obviously be done by taking $c^2 = n+2$ and $A = \Phi^{-1}$. Thus the ellipsoid

$$X \Phi^{-1} X' = n+2$$

has the required property. Suppose X has mean (μ_1, \dots, μ_n) and variance-covariance matrix Φ . Let $Y_i = X_i - \mu_i$, $Y = (Y_1, \dots, Y_n)$. Then the required ellipsoid is

$$Y \Phi^{-1} Y' = n+2 \quad .$$

This is called the ellipsoid of concentration.

In some cases the ellipsoid of concentration, with the parameters replaced by sample estimates, may serve as a simple descriptive statistic.